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Asymptotic Expansions in the Conditional Central Limit Theorem

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Let X_n, $n \in \mathbb{N}$, be i.i.d. with mean 0, variance 1, and $E(|X_n|^r) < \infty$ for some $r \geqslant 3$. Assume that Cramér's condition is fulfilled. We prove that the conditional probabilities $P(1/\sqrt{n} \sum_{i=1}^{n} X_i \leqslant t \mid B)$ can be approximated by a modified Edgeworth expansion up to order $o(1/n^{(r-2)/2})$, if the distances of the set B from the σ-fields $\sigma(X_1, ..., X_n)$ are of order $O(1/n^{(r-2)/2} (\lg n)^{\beta})$, where $\beta < -(r-2)/2$ for $r \notin \mathbb{N}$ and $\beta < -r/2$ for $r \in \mathbb{N}$. An example shows that if we replace $\beta < -(r-2)/2$ by $\beta = -(r-2)/2$ for $r \notin \mathbb{N}$ ($\beta < -r/2$ by $\beta = -r/2$ for $r \in \mathbb{N}$) we can only obtain the approximation order $O(1/n^{(r-2)/2})$ for $r \notin \mathbb{N}$ ($O(\lg \lg n / n^{(r-2)/2})$ for $r \in \mathbb{N}$). © 1990 Academic Press, Inc.

1. INTRODUCTION AND NOTATIONS

Let X_n, $n \in \mathbb{N}$, be a sequence of i.i.d. real valued random variables with mean 0 and variance 1. Put $S_n = \sum_{i=1}^{n} X_i$ and $S_n^* = 1/\sqrt{n} \sum_{i=1}^{n} X_i$. Denote by $d(B, \sigma(X_1, ..., X_n)) := \inf\{P(B \triangle B_n) : B_n \in \sigma(X_1, ..., X_n)\}$ the distance of the set B from the σ-field $\sigma(X_1, ..., X_n)$. In this paper we look for Edgeworth expansions of the conditional probabilities $P(S_n^* \leqslant t \mid B)$. If $E(|X_1|^r) < \infty$ for some $r \geqslant 3$ and if Cramér's condition is fulfilled, i.e., $\overline{\lim}_{|t| \to \infty} |E(e^{itX_1})| < 1$, then we have (for $B = \Omega$) the well-known expansion

$$\sup_{t \in \mathbb{R}} \left| P(S_n^* \leqslant t) - \Phi(t) - \varphi(t) \sum_{i=1}^{[r]-2} \frac{1}{n^{i/2}} Q_i(t) \right| = o\left(\frac{1}{n^{(r-2)/2}}\right)$$

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(see, e.g., Theorem 2, p. 168 of Petrov [6]). Here Φ denotes the standard normal distribution function and φ its density.

$Q_i(t)$ are the classical polynomials and $[x] = \max\{n \in \mathbb{N} : n \leqslant x\}$. For more general sets B there exists only one expansion result (see [4]). This result deals with the case $r = 4$ and uses one correcting term. It was shown in [4] that $d(B, \sigma(X_1, ..., X_n)) = O(1/n(\lg n)^\beta)$ for some $\beta < -2$ implies that

$$\sup_{t \in \mathbb{R}} \left| P(S_n^* \leqslant t \mid B) - \Phi(t) - \varphi(t) \frac{\hat{Q}_{1,B}(t)}{n^{1/2}} \right| = O\left(\frac{1}{n}\right)$$

with $\hat{Q}_{1,B}(t) = Q_1(t) - a$, where a is a constant depending on B and the distribution of X_1.

In this paper we give higher order asymptotic expansions for $P(S_n^* \leqslant t \mid B)$. We prove that

$$d(B, \sigma(X_1, ..., X_n)) = O\left(\frac{1}{n^{(r-2)/2}} (\lg n)^\beta\right)$$

implies that there exist polynomials $\hat{Q}_{i,B}(t)$ such that uniformly in $t \in \mathbb{R}$,

$$\left| P(S_n^* \leqslant t \mid B) - \Phi(t) - \varphi(t) \sum_{i=1}^{[r]-2} \frac{\hat{Q}_{i,B}(t)}{n^{i/2}} \right|$$

$$= \begin{cases} o\left(\dfrac{1}{n^{(r-2)/2}}\right), & r \notin \mathbb{N}, \quad \beta < -\dfrac{r-2}{2} \\[2ex] O\left(\dfrac{(\lg n)^{\beta + (r-2)/2}}{n^{(r-2)/2}}\right), & r \notin \mathbb{N}, \quad \beta \geqslant -\dfrac{r-2}{2} \\[2ex] o\left(\dfrac{1}{n^{(r-2)/2}}\right), & r \in \mathbb{N}, \quad \beta < -r/2 \\[2ex] O\left(\dfrac{\lg \lg n}{n^{(r-2)/2}}\right), & r \in \mathbb{N}, \quad \beta = -r/2 \\[2ex] O\left(\dfrac{(\lg n)^{\beta + r/2}}{n^{(r-2)/2}}\right), & r \in \mathbb{N}, \quad \beta > -r/2 \end{cases}$$

(see Theorem 1 with $g = 1_B$).

This result shows a surprising difference between the cases $r \in \mathbb{N}$ and $r \notin \mathbb{N}$. Nevertheless all approximation orders are optimal (see Example 2).

2. The Results

In this section we present our results, postponing the proofs until Section 3.

If g is a measurable function we denote by

$$d_1(g, \sigma(X_1, ..., X_n)) := \inf\{E(|g-h|): h \text{ is } \sigma(X_1, ..., X_n) \text{ measurable}\}$$

the $\| \|_1$-distance of g from the subspace of all integrable $\sigma(X_1, ..., X_n)$ measurable functions. We write $E(S_n^* \leqslant t, g)$ instead of $E(g \cdot 1_{\{S_n^* \leqslant t\}})$.

The following theorem is the main result of this paper.

Since $\varphi(t)(1/n^{([r]-2)/2})\, Q_{[r]-2, g}(t) = O_n(r, \beta)$ for the last two cases of this theorem (i.e., for $r \in \mathbb{N}$, $\beta \geqslant -r/2$) we omit in these cases the last term of the expansion. Hence we consider in these cases the expansion up to the $(r-3)$ th term only. Observe that all convergence orders $O_n(r, \beta)$ are optimal (see Example 2).

THEOREM 1. *Let $r \geqslant 3$ and let $X_n, n \in \mathbb{N}$, be i.i.d. with $E(X_1) = 0$, $E(X_1^2) = 1$, and $E(|X_1|^r) < \infty$. Assume that Cramér's condition is fulfilled. Let g be a bounded measurable function, let $\beta \in \mathbb{R}$, and assume that*

$$d_1(g, \sigma(X_1, ..., X_n)) = O\left(\frac{1}{n^{(r-2)/2}} (\lg n)^{\beta}\right). \tag{$*$}$$

Then there exist polynomials $Q_{i,g}(t)$ (the coefficients depend on g and on the distribution of X_1) such that

$$\sup_{t \in \mathbb{R}} \left| E(S_n^* \leqslant t, g) - \Phi(t)\, E(g) - \varphi(t) \sum_{i=1}^{j(r, \beta)} \frac{1}{n^{i/2}} Q_{i,g}(t) \right| = O_n(r, \beta),$$

where

$$j(r, \beta) = \begin{cases} [r]-2, & \text{if } r \notin \mathbb{N} \text{ or } r \in \mathbb{N}, \quad \beta < -r/2 \\ r-3, & \text{if } r \in \mathbb{N}, \quad \beta \geqslant -r/2 \end{cases}$$

and

$$O_n(r, \beta) = \begin{cases} o\left(\dfrac{1}{n^{(r-2)/2}}\right), & \text{if } r \notin \mathbb{N}, \quad \beta < -\dfrac{r-2}{2} & \text{(i)} \\ O\left(\dfrac{(\lg n)^{\beta + (r-2)/2}}{n^{(r-2)/2}}\right), & \text{if } r \notin \mathbb{N}, \quad \beta \geqslant -\dfrac{r-2}{2} & \text{(ii)} \\ o\left(\dfrac{1}{n^{(r-2)/2}}\right), & \text{if } r \in \mathbb{N}, \quad \beta < -r/2 & \text{(iii)} \\ O\left(\dfrac{\lg \lg n}{n^{(r-2)/2}}\right), & \text{if } r \in \mathbb{N}, \quad \beta = -r/2 & \text{(iv)} \\ O\left(\dfrac{(\lg n)^{\beta + r/2}}{n^{(r-2)/2}}\right), & \text{if } r \in \mathbb{N}, \quad \beta > -r/2. & \text{(v)} \end{cases}$$

Remark. The polynomials $Q_{i,g}(t)$ of Theorem 1 can be computed alon the lines of the proof of Theorem 1. We have, e.g.,

$$Q_{1,g}(t) = Q_1(t)\, E(g) - a_1$$

$$Q_{2,g}(t) = Q_2(t)\, E(g) + (\tfrac{1}{2} E(X_1^3)\, a_1 - \tfrac{1}{2} a_2)\, t - \tfrac{1}{6} a_1 E(X_1^3)\, t^3,$$

where a_1, a_2 are constants depending on g and on the distribution of X For $Q_{1,g}(t)$ see also Theorem 1 of [4].

The following example shows that the approximation orders give in Theorem 1 are optimal. It is well known that even if $g = 1_\Omega$– whence $d_1(g, \sigma(X_1, ..., X_n)) \equiv 0$—the approximation orders $o(1/n^{(r-2)/2})$ (Theorem 1 (i.e., case (i) and case (iii)) cannot be improved. Therefoı Example 2 deals with the remaining three cases. Always we choos $g = 1_B$ with a suitable set B. Observe that $d_1(1_B, \sigma(X_1, ..., X_n)) \leqslant$ $d(B, \sigma(X_1, ..., X_n))$ (this can be shown, e.g., by using the Fubini Theorem

EXAMPLE 2. Let X_n, $n \in \mathbb{N}$, be i.i.d. $N(0, 1)$-distributed. Let $r \geqslant 3$, $\beta \in \mathbb{F}$ Then there exist $B \in \sigma(X_n : n \in \mathbb{N})$ and $t_0 \in \mathbb{R}$, $c > 0$, such that

$$d(B, \sigma(X_1, ..., X_n)) = O\left(\frac{1}{n^{(r-2)/2}} (\lg n)^\beta\right) \qquad (*$$

and

$$\left| E(S_n^* \leqslant t_0, B) - \Phi(t_0)\, P(B) - \varphi(t_0) \sum_{i=1}^{j} \frac{1}{n^{i/2}} Q_{i,B}(t_0) \right| \geqslant c\delta_n$$

for infinitely many $n \in \mathbb{N}$, where

$$j = j(r) = \begin{cases} [r] - 2, & r \notin \mathbb{N} \\ r - 3, & r \in \mathbb{N} \end{cases}$$

and

$$\delta_n = \delta_n(r, \beta) = \begin{cases} \dfrac{(\lg n)^{\beta + (r-2)/2}}{n^{(r-2)/2}}, & \text{if } r \notin \mathbb{N}, \quad \beta \geqslant -\dfrac{r-2}{2} \\[2ex] \dfrac{\lg \lg n}{n^{(r-2)/2}}, & \text{if } r \in \mathbb{N}, \quad \beta = -r/2 \\[2ex] \dfrac{(\lg n)^{\beta + r/2}}{n^{(r-2)/2}}, & \text{if } r \in \mathbb{N}, \quad \beta > -r/2. \end{cases}$$

Here $Q_{i,B}(t) = Q_{i,1_B}(t)$ are the polynomials of Theorem 1.

3. Proof of the Results

To prevent the proof of Theorem 1 from becoming too lengthy we try to unify the proof as far as possible for the rather different types of approximation orders $O_n(r, \beta)$. Some lemmas which are needed for the proofs of Theorem 1 and Example 2 are given at the end of this section.

Proof of Theorem 1. Let $j \in \mathbb{N} \cup \{0\}$ be fixed. We prove the result for pairs (r, β) with $j(r, \beta) = j$. For $j = 0$, we have $r = 3$, $\beta \geqslant -\frac{3}{2}$ and the result is part of Theorem 4 of [3]. We assume therefore that $j \geqslant 1$. We need some conventions and notations. Throughout the proof we use the symbol c to denote a general constant which may depend on r, β, and the distribution of X_1. Put $\mathbb{N}_1 = \{2^i : i \in \mathbb{N}\}$, $N_n = \{v \in \mathbb{N}_1 : v \leqslant n/\lg n\}$, and define $k(n) = \max N_n$, $n \geqslant 2$. Let g be a bounded and measurable function, fulfilling condition $(*)$ of Theorem 1. Choose $\sigma(X_1, ..., X_n)$ measurable functions g_n with $E(|g - g_n|) = d_1(g, \sigma(X_1, ..., X_n))$. Put $h_2 = g_2$ and $h_v = g_v - g_{v/2}$ for each $v \in \mathbb{N}_1$, $v \geqslant 4$. Then we obtain by assumption $(*)$

$$E(|h_v|) \leqslant c \frac{(\lg v)^\beta}{v^{(r-2)/2}}, \qquad v \in \mathbb{N}_1. \tag{1}$$

We show first two relations which are essential tools for the proof:

(A) $$\frac{1}{n^{l+\tau/2}} \sum_{k(n) \leqslant v \in \mathbb{N}_1} v^l E(|h_v| \, |S_v|^\tau) = O_n(r, \beta)$$

if $l + \tau/2 \leqslant j/2$, $l \geqslant 0$, $\tau \geqslant 0$, *and* $l, \tau \in \mathbb{R}$.

(B) $$\frac{1}{n^{l+\tau/2}} \sum_{v \in N_n} v^l E(|h_v| \, |S_v|^\tau) = O_n(r, \beta)$$

if $l + \tau/2 \geqslant (j+1)/2$, $l \geqslant 0$, $0 \leqslant \tau < r$, *and* $l, \tau \in \mathbb{R}$.

Ad (A). If $v \geqslant 2$, $0 < \tau < r$, we have by Lemma 4 and (1) for each $\gamma \geqslant \frac{1}{2}$

$$\begin{aligned} E(|h_v| \, |S_v|^\tau) &\leqslant cE(|S_v|^\tau 1_{\{|S_v^*| \geqslant \sqrt{r-1}\,(\lg v)^\gamma\}}) + cv^{\tau/2} (\lg v)^{\gamma\tau} E(|h_v|) \\ &\leqslant cv^{\tau/2 - (r-2)/2} ((\lg v)^{\gamma(\tau - r)} + (\lg v)^{\gamma\tau + \beta}). \end{aligned} \tag{2}$$

For $\tau = 0$, (2) follows from (1).

Relation (2) implies

$$\begin{aligned} H(n) &:= \frac{1}{n^{l+\tau/2}} \sum_{k(n) \leqslant v \in \mathbb{N}_1} v^l E(|h_v| \, |S_v|^\tau) \\ &\leqslant c \frac{1}{n^{l+\tau/2}} \sum_{k(n) \leqslant v \in \mathbb{N}_1} v^{l+\tau/2-(r-2)/2} ((\lg v)^{\gamma(\tau-r)} + (\lg v)^{\gamma\tau+\beta}). \end{aligned} \tag{3}$$

We consider at first the case $l+\tau/2<(r-2)/2$. As

$$\sum_{k(n)\leqslant\nu\in\mathbb{N}_1}\frac{1}{\nu^{\varepsilon}}(\lg\nu)^{\delta}\leqslant\frac{c}{n^{\varepsilon}}(\lg n)^{\varepsilon+\delta}\qquad\text{if}\quad\varepsilon>0$$

we obtain from (3) with $\gamma=\frac{1}{2}$

$$H(n)\leqslant c\,\frac{1}{n^{(r-2)/2}}(\lg n)^{(r-2)/2-(l+\tau/2)}\left[(\lg n)^{(\tau-r)/2}+(\lg n)^{\tau/2+\beta}\right]$$

$$=c\,\frac{1}{n^{(r-2)/2}}\left[\frac{1}{(\lg n)^{1+l}}+(\lg n)^{\beta+(r-2)/2-l}\right]=O_n(r,\beta).$$

As $l+\tau/2\leqslant j/2\leqslant([r]-2)/2\leqslant(r-2)/2$ it remains to consider the case $l+\tau/2=(r-2)/2$. Hence $j=j(r,\beta)=r-2$, whence $r\in\mathbb{N}$, $\beta<-r/2$. Consequently there exists γ with $\frac{1}{2}<\gamma<-(\beta+1)/j=-(\beta+1)/(r-2)$. Then $\gamma(\tau-r)<-1$ and $\gamma\tau+\beta<-1$, and (3) implies

$$H(n)=o\left(\frac{1}{n^{(r-2)/2}}\right)=O_n(r,\beta).$$

Ad (B). We obtain from (2) for each $\gamma\geqslant\frac{1}{2}$,

$$L(n)=\frac{1}{n^{l+\tau/2}}\sum_{\nu\in N_n}\nu^{l}E(|h_\nu|\,|S_\nu|^{\tau})$$

$$\leqslant c\,\frac{1}{n^{l+\tau/2}}\sum_{\nu\in N_n}\nu^{l+\tau/2-(r-2)/2}((\lg\nu)^{\gamma(\tau-r)}+(\lg\nu)^{\gamma\tau+\beta}).\tag{4}$$

We consider the three cases $l+\tau/2\lesseqgtr(r-2)/2$:

(i) As $l+\tau/2\geqslant(j+1)/2\geqslant(r-2)/2$, $l+\tau/2<(r-2)/2$ is impossible.

(ii) If $l+\tau/2>(r-2)/2$, apply (4) with $\gamma=\frac{1}{2}$. Then we obtain using Lemma 5

$$L(n)\leqslant c\,\frac{1}{n^{l+\tau/2}}n^{l+\tau/2-(r-2)/2}\left(\frac{1}{(\lg n)^{1+l}}+(\lg n)^{\beta+(r-2)/2-l}\right)$$

$$\leqslant c\,\frac{1}{n^{(r-2)/2}}\left(\frac{1}{\lg n}+(\lg n)^{\beta+(r-2)/2}\right)=O_n(r,\beta).$$

(iii) Finally let $l+\tau/2=(r-2)/2$.

Hence $(r-2)/2=l+\tau/2\geqslant(j+1)/2$, i.e., $j\leqslant r-3$, whence $r\in\mathbb{N}$ and $\beta\geqslant-r/2$.

Applying (4) with $\gamma=\frac{1}{2}$ we obtain

$$L(n)\leqslant c\,\frac{1}{n^{(r-2)/2}}\sum_{\nu\in N_n}\left(\frac{1}{\lg\nu}+(\lg\nu)^{\beta+(r-2)/2}\right). \tag{5}$$

By Lemma 5 we have $\sum_{\nu\in N_n}1/\lg\nu=O(\lg\lg n)$ and

$$\sum_{\nu\in N_n}(\lg\nu)^{\beta+(r-2)/2}=\begin{cases}O(\lg\lg n), & \text{if } \beta=-r/2\\ O((\lg n)^{\beta+r/2}), & \text{if } \beta>-r/2.\end{cases}$$

Hence (5) implies $L_n=O_n(r,\beta)$. Thus (A) and (B) are proven.

Using $1-\Phi(\sqrt{r\lg n})=o(1/n^{(r-2)/2})$ and similar methods as in the proof of Theorem 1 of [4], it suffices to construct polynomials $Q_{i,g}(t)$, $i=1,\ldots,j$, such that

$$\sup_{|t|\leqslant\sqrt{r\lg n}}\left|E(S_n^*\leqslant t,g)-\Phi(t)\,E(g)-\varphi(t)\sum_{i=1}^{j}\frac{1}{n^{i/2}}Q_{i,g}(t)\right|=O_n(r,\beta). \tag{6}$$

Since $g=g-g_{k(n)}+\sum_{\nu\in N_n}h_\nu$, we obtain by assumption $(*)$

$$\sup_{t\in\mathbb{R}}\left|E(S_n^*\leqslant t,g)-\sum_{\nu\in N_n}E(S_n^*\leqslant t,h_\nu)\right|$$

$$\leqslant E(|g-g_{k(n)}|)=O\left(\frac{1}{(k(n))^{(r-2)/2}}(\lg k(n))^{\beta}\right)$$

$$=O\left(\frac{1}{n^{(r-2)/2}}(\lg n)^{\beta+(r-2)/2}\right)=O_n(r,\beta).$$

Hence it suffices to prove that

$$\sup_{|t|\leqslant\sqrt{r\lg n}}\left|\sum_{\nu\in N_n}E(S_n^*\leqslant t,h_\nu)-\Phi(t)\,E(g)-\varphi(t)\sum_{i=1}^{j}\frac{1}{n^{i/2}}Q_{i,g}(t)\right|$$

$$=O_n(r,\beta). \tag{7}$$

Let F_n be the distribution function of S_n^* and let

$$K_{n,j}(t)=\Phi(t)+\varphi(t)\sum_{i=1}^{j}\frac{1}{n^{i/2}}Q_i(t)$$

be the classical asymptotic expansions. Put $D_{n,j}=F_n-K_{n,j}$.

We prove three properties which imply our assertion as we see later:

(P1) $$\sup_{t\in\mathbb{R}}\left|\sum_{\nu\in N_n}\int h_\nu(\omega)\,D_{n-\nu,j}\left(\sqrt{\frac{n}{n-\nu}}\,t-\frac{1}{\sqrt{n-\nu}}S_\nu(\omega)\right)P(d\omega)\right|$$

$$=O_n(r,\beta)$$

(P2) $$\sup_{t\in\mathbb{R}}\left|\sum_{\nu\in N_n}\int h_\nu(\omega)\,K_{n-\nu,j}(t)\,P(d\omega)\right.$$
$$\left.-\Phi(t)\,E(g)-\varphi(t)\sum_{i=1}^{j}\frac{1}{n^{i/2}}\,Q_{i,g}^{(1)}(t)\right|=O_n(r,\beta)$$

(P3) $$\sup_{|t|\leqslant\sqrt{r\lg n}}\left|\sum_{\nu\in N_n}\int h_\nu(\omega)\left[K_{n-\nu,j}\left(\sqrt{\frac{n}{n-\nu}}\,t-\frac{1}{\sqrt{n-\nu}}\,S_\nu(\omega)\right)\right.\right.$$
$$\left.\left.-K_{n-\nu,j}(t)\right]P(d\omega)-\varphi(t)\sum_{i=1}^{j}\frac{1}{n^{i/2}}\,Q_{i,g}^{(2)}(t)\right|=O_n(r,\beta)$$

with suitable polynomials $Q_{i,g}^{(1)}(t)$, $Q_{i,g}^{(2)}(t)$.

Ad (P1). Since Cramér's condition is fulfilled, we have by the classical asymptotic expansion (see [6, Theorem 2, p. 168]) that

$$\sup_{y\in\mathbb{R}}|D_{n-\nu,j}(y)|\leqslant\begin{cases}c\,\dfrac{\varepsilon_{n-\nu}}{(n-\nu)^{(r-2)/2}}, & \text{if } j=[r]-2\\ c\,\dfrac{1}{(n-\nu)^{(r-2)/2}}, & \text{if } j=r-3,\end{cases}\tag{8}$$

where $\varepsilon_m\rightarrow_{m\in\mathbb{N}}0$. Since $n-\nu\geqslant n/2$ for all $\nu\in N_n$ (if $\lg n\geqslant 2$), (8) implies

$$\sup_{\nu\in N_n,\,y\in\mathbb{R}}|D_{n-\nu,j}(y)|=O_n(r,\beta).\tag{9}$$

Let A_n be the expression occurring in (P1). Then (9) and (1) imply

$$A_n\underset{(9)}{\leqslant}\sum_{\nu\in N_n}E(|h_\nu|)\,O_n(r,\beta)\underset{(1)}{=}O_n(r,\beta).$$

Ad (P2). By definition of $K_{n-\nu,j}$, we have

$$\sum_{\nu\in N_n}\int h_\nu(\omega)\,K_{n-\nu,j}(t)\,P(d\omega)$$
$$=\Phi(t)\,E(g_{k(n)})+\varphi(t)\sum_{\nu\in N_n}\left(E(h_\nu)\sum_{i=1}^{j}\frac{1}{(n-\nu)^{i/2}}\,Q_i(t)\right).\tag{10}$$

For $\nu\in N_n$, $n\in\mathbb{N}$, and $i\leqslant j$, we have

$$\frac{1}{(n-\nu)^{i/2}}=\frac{1}{n^{i/2}}\left(\sum_{l=0}^{j}\binom{-i/2}{l}\left(-\frac{\nu}{n}\right)^l+O\left(\left(\frac{\nu}{n}\right)^{j+1}\right)\right),$$
$$\text{where } O\left(\frac{\nu}{n}\right)\leqslant c\,\frac{\nu}{n}.$$

Hence (10) implies

$$\sum_{\nu \in N_n} K_{n-\nu,j}(t)\, E(h_\nu)$$
$$= \Phi(t)\, E(g_{k(n)})$$
$$+ \varphi(t) \sum_{i=1}^{j} \sum_{l=0}^{j} \binom{-i/2}{l} \frac{1}{n^{i/2+l}} \sum_{\nu \in N_n} (-\nu)^l\, E(h_\nu)\, Q_i(t)$$
$$+ \varphi(t) \sum_{i=1}^{j} \frac{1}{n^{i/2}} \sum_{\nu \in N_n} O\left(\left(\frac{\nu}{n}\right)^{j+1}\right) E(h_\nu)\, Q_i(t).$$

As $E(g_{k(n)}) = E(g) + O_n(r, \beta)$, (P2) is shown if we prove that for $1 \leqslant i \leqslant j$, $0 \leqslant l \leqslant j$,

$$\frac{1}{n^{i/2+l}} \sum_{\nu \in N_n} (-\nu)^l\, E(h_\nu) = \frac{c}{n^{i/2+l}} + O_n(r, \beta) \quad \text{for} \quad i/2 + l \leqslant j/2 \tag{11}$$

$$\frac{1}{n^{i/2+l}} \sum_{\nu \in N_n} \nu^l E(|h_\nu|) = O_n(r, \beta) \quad \text{for} \quad i/2 + l > j/2. \tag{12}$$

Ad (11). As $i/2 + l \leqslant j/2$ and $i \geqslant 1$, we have $l < j/2$. Hence (1) applied to $\tau = 0$ yields that $\sum_{\nu \in \mathbb{N}_1} \nu^l E(|h_\nu|) < \infty$. Put $c = \sum_{\nu \in \mathbb{N}_1} (-\nu)^l E(h_\nu)$. Then (A) applied to $\tau = 0$ yields

$$\left| \frac{1}{n^{i/2+l}} \left(\sum_{\nu \in N_n} (-\nu)^l\, E(h_\nu) - c \right) \right| \leqslant \frac{1}{n^{i/2+l}} \sum_{k(n) \leqslant \nu \in \mathbb{N}_1} \nu^l E(|h_\nu|)$$
$$\leqslant \frac{1}{n^l} \sum_{k(n) \leqslant \nu \in \mathbb{N}_1} \nu^l E(|h_\nu|) = O_n(r, \beta).$$

Ad (12). (B) applied to $\tau = 0$ and $i/2 + l$ instead of l yields

$$\frac{1}{n^{i/2+l}} \sum_{\nu \in N_n} \nu^l E(|h_\nu|) \leqslant \frac{1}{n^{i/2+l}} \sum_{\nu \in N_n} \nu^{i/2+l} E(|h_\nu|) = O_n(r, \beta).$$

Ad (P3). Let $u := u_{t,n,\nu}(\omega) = \sqrt{n/(n-\nu)}\, t - (1/\sqrt{n-\nu})\, S_\nu(\omega) = t(f(\nu/n) + 1) - (1/\sqrt{n-\nu})\, S_\nu(\omega)$, where $f(x) = (1-x)^{-1/2} - 1 = \sum_{p=1}^{\infty} \binom{-1/2}{p} (-x)^p \leqslant cx$ for $0 \leqslant x \leqslant \frac{1}{2}$.

Hence we have for $\nu \in N_n$, $n \in \mathbb{N}$,

$$|u_{t,n,\nu}(\omega) - t| \leqslant c \left(|t| \frac{\nu}{n} + \frac{1}{\sqrt{n}} |S_\nu(\omega)| \right),$$

whence

$$|u_{t,n,\nu}(\omega)-t|^{j+1}\leqslant c\left(|t|^{j+1}\left(\frac{\nu}{n}\right)^{j+1}+\frac{1}{n^{(j+1)/2}}|S_\nu(\omega)|^{j+1}\right). \tag{13}$$

By the Taylor expansion we have

$$\begin{aligned}K_{n-\nu,j}(u)-K_{n-\nu,j}(t)=&\sum_{\lambda=1}^{j}\frac{1}{\lambda!}K_{n-\nu,j}^{(\lambda)}(t)(u-t)^\lambda\\&+\frac{1}{(j+1)!}K_{n-\nu,j}^{(j+1)}(\xi)(u-t)^{j+1}\end{aligned} \tag{14}$$

with $\xi=\xi_{t,n,\nu}(\omega)\in[u_{t,n,\nu}(\omega),t]$.

According to (14), property (P3) is shown if we prove that

$$\begin{aligned}B_n:=&\sup_{|t|\leqslant\sqrt{r\lg n}}\left|\sum_{\nu\in N_n}\int h_\nu(\omega)(u_{t,n,\nu}(\omega)-t)^{j+1}\right.\\&\left.\times K_{n-\nu,j}^{(j+1)}(\xi_{t,n,\nu}(\omega))\,P(d\omega)\right|\\=&O_n(r,\beta)\end{aligned} \tag{15}$$

and that for each $\lambda=1,\dots,j$ there holds uniformly in $|t|\leqslant\sqrt{r\lg n}$

$$\begin{aligned}&\sum_{\nu\in N_n}K_{n-\nu,j}^{(\lambda)}(t)\int(u_{t,n,\nu}(\omega)-t)^\lambda h_\nu(\omega)\,P(d\omega)\\&\quad=\varphi(t)\sum_{p=1}^{j}\frac{1}{n^{p/2}}Q_{p,g,\lambda}(t)+O_n(r,\beta)\end{aligned} \tag{16}$$

with suitable polynomials $Q_{p,g,\lambda}(t)$.

Ad (15). As $\sup\{|K_{n-\nu,j}^{(j+1)}(\xi)|:\xi\in\mathbb{R},\ n\in\mathbb{N},\ \nu\in N_n\}<\infty$, we obtain from (13) that

$$\begin{aligned}B_n\leqslant&\, c\sup_{|t|\leqslant\sqrt{r\lg n}}\sum_{\nu\in N_n}\int|h_\nu(\omega)|\,|u_{t,n,\nu}(\omega)-t|^{j+1}P(d\omega)\\\leqslant&\, c\frac{(\lg n)^{(j+1)/2}}{n^{j+1}}\sum_{\nu\in N_n}\nu^{j+1}E(|h_\nu|)\\&+\frac{c}{n^{(j+1)/2}}\sum_{\nu\in N_n}E(|h_\nu|\,|S_\nu|^{j+1}).\end{aligned} \tag{13}$$

Hence by (1) and (B)

$$B_n \leqslant c \frac{(\lg n)^{(j+1)/2}}{n^{j+1}} \sum_{\nu \in N_n} \frac{\nu^{j+1}}{\nu^{(r-2)/2}} (\lg \nu)^{\beta} + O_n(r, \beta).$$

Consequently by Lemma 5

$$B_n \leqslant c \frac{(\lg n)^{(j+1)/2}}{n^{(r-2)/2}} (\lg n)^{\beta - (j+1) + (r-2)/2} + O_n(r, \beta) = O_n(r, \beta).$$

Thus we have (15).

Ad (16). Let $\lambda \in \{1, ..., j\}$ be fixed. We have with suitable polynomials $\hat{Q}_i(t)$ that

$$\begin{aligned} K^{(\lambda)}_{n-\nu,j}(t) &= \Phi^{(\lambda)}(t) + \sum_{i=1}^{j} \frac{1}{(n-\nu)^{i/2}} (\varphi \cdot Q_i)^{(\lambda)}(t) \\ &= \varphi(t) \sum_{i=0}^{j} \frac{1}{(n-\nu)^{i/2}} \hat{Q}_i(t). \end{aligned} \tag{17}$$

Furthermore we have by definition of $u_{t,n,\nu}(\omega)$ and $f(x)$ that

$$(u_{t,n,\nu}(\omega) - t)^{\lambda} = \sum_{\varepsilon=0}^{\lambda} \binom{\lambda}{\varepsilon} t^{\varepsilon} f^{\varepsilon}\left(\frac{\nu}{n}\right) (-1)^{\lambda-\varepsilon} \frac{1}{(n-\nu)^{(\lambda-\varepsilon)/2}} S_{\nu}^{\lambda-\varepsilon}(\omega). \tag{18}$$

According to (17) and (18), relation (16) is shown if we prove that for each $0 \leqslant i \leqslant j$, $0 \leqslant \varepsilon \leqslant \lambda$ uniformly in $|t| \leqslant \sqrt{r \lg n}$,

$$\begin{aligned} &\varphi(t) \hat{Q}_i(t) \binom{\lambda}{\varepsilon} (-1)^{\lambda-\varepsilon} t^{\varepsilon} \sum_{\nu \in N_n} \frac{f^{\varepsilon}(\nu/n)}{(n-\nu)^{(i+\lambda-\varepsilon)/2}} E(h_{\nu} S_{\nu}^{\lambda-\varepsilon}) \\ &\quad = \varphi(t) \sum_{p=1}^{j} \frac{1}{n^{p/2}} R_p(t) + O_n(r, \beta) \end{aligned}$$

with suitable polynomials $R_p(t) = R_{p,i,\varepsilon,\lambda,g}(t)$.

We have

$$f^{\varepsilon}\left(\frac{\nu}{n}\right) \frac{1}{(n-\nu)^{(\lambda-\varepsilon+i)/2}} = \frac{1}{n^{(\lambda-\varepsilon+i)/2}} \frac{(1-\sqrt{1-\nu/n})^{\varepsilon}}{(1-\nu/n)^{(\lambda+i)/2}}.$$

By Taylor expansion we furthermore have

$$q_{\varepsilon}(x) := q_{\varepsilon,\lambda,i}(x) := \frac{(1-\sqrt{1-x})^{\varepsilon}}{(1-x)^{(\lambda+i)/2}} = \sum_{l=0}^{j} \frac{q_{\varepsilon}^{(l)}(0)}{l!} x^l + O(x^{j+1}).$$

Hence for $\nu \in N_n$, $n \in \mathbb{N}$,

$$f^{\varepsilon}\left(\frac{\nu}{n}\right)\frac{1}{(n-\nu)^{(\lambda-\varepsilon+i)/2}}=\frac{1}{n^{(\lambda-\varepsilon+i)/2}}\left[\sum_{l=0}^{j}\frac{q_{\varepsilon}^{(l)}(O)}{l!}\left(\frac{\nu}{n}\right)^{l}+O\left(\left(\frac{\nu}{n}\right)^{j+1}\right)\right].$$

Observe that $q_{\varepsilon}^{(0)}(0)=0$ if $\varepsilon>0$. Consequently (19) is shown if we prove that

$$\frac{1}{n^{(\lambda-\varepsilon+i)/2+l}}\sum_{\nu\in N_n}\nu^{l}E(h_{\nu}S_{\nu}^{\lambda-\varepsilon})=\frac{c}{n^{(\lambda-\varepsilon+i)/2+l}}+O_n(r,\beta) \tag{20}$$

for $\frac{1}{2}\leqslant(\lambda-\varepsilon+i)/2+l\leqslant j/2$ and

$$\frac{1}{n^{(\lambda-\varepsilon+i)/2+l}}\sum_{\nu\in N_n}\nu^{l}E(|h_{\nu}|\,|S_{\nu}|^{\lambda-\varepsilon})=O_n(r,\beta) \tag{21}$$

for $(\lambda-\varepsilon+i)/2+l>j/2$. Relation (20) follows from (A) with $c=\sum_{\nu\in\mathbb{N}_1}\nu^{l}E(h_{\nu}S_{\nu}^{\lambda-\varepsilon})$. Relation (21) follows from a slight modification of (B). Thus (P3) is shown.

Now it remains to show that (P1)–(P3) imply the assertion, i.e., we have to prove (7).

Since for $\nu<n$ the function $\omega\to F_{n-\nu}(\sqrt{n/(n-\nu)}\,t-(1/\sqrt{n-\nu})\,S_{\nu}(\omega))$ is a version of $P(S_n^*\leqslant t\mid X_1,\ldots,X_{\nu})$ and since h_{ν} is $\sigma(X_1,\ldots,X_{\nu})$-measurable we obtain that

$$E(S_n^*\leqslant t,h_{\nu})=\int h_{\nu}(\omega)\,F_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\,t-\frac{1}{\sqrt{n-\nu}}\,S_{\nu}(\omega)\right)P(d\omega).$$

Hence

$$\begin{aligned}&\sum_{\nu\in N_n}E(S_n^*\leqslant t,h_{\nu})\\&\quad=\sum_{\nu\in N_n}\int h_{\nu}(\omega)\,D_{n-\nu,j}\left(\sqrt{\frac{n}{n-\nu}}\,t-\frac{1}{\sqrt{n-\nu}}\,S_{\nu}(\omega)\right)P(d\omega)\\&\qquad+\sum_{\nu\in N_n}\int h_{\nu}(\omega)\left[K_{n-\nu,j}\left(\sqrt{\frac{n}{n-\nu}}\,t-\frac{1}{\sqrt{n-\nu}}\,S_{\nu}(\omega)\right)-K_{n-\nu,j}(t)\right]P(d\omega)\\&\qquad+\sum_{\nu\in N_n}\int h_{\nu}(\omega)\,K_{n-\nu,j}(t)\,P(d\omega).\end{aligned}$$

Thus (P1)–(P3) imply (7) and hence the assertion.

Proof of Example 2. For the case $r=3$ see Example 5 of [2] with $h(n)\equiv 1$ if $\beta=-\frac{3}{2}$ and $h(n)=(\lg n)^{\beta+r/2}$ if $\beta>-r/2$.

Therefore we assume $r > 3$. The concept for all three cases of this example is the following: Let $t_0 \in \mathbb{R}$ and $c_0 \in (0, 1]$ be the constants of Lemma 3 and put $k(n) := [c_0(n/\lg n)]$. We construct a subsequence $\hat{\mathbb{N}} \subset \mathbb{N}$ and disjoint sets $B_\nu \in \sigma(X_1, ..., X_\nu)$, $\nu \in \mathbb{N}$, with the following properties:

(P1) $B_\nu \subset \{\sqrt{\lg \nu}/2 \leqslant S_\nu^* \leqslant \sqrt{\lg \nu}\}, \qquad \nu \in \mathbb{N}$

(P2) $\sum_{\nu > n} P(B_\nu) = O\left(\frac{1}{n^{(r-2)/2}} (\lg n)^\beta\right)$

(P3) $\sum_{\nu > k(n)} P(B_\nu) = o(\delta_n), \qquad n \in \hat{\mathbb{N}}$

(P4) $\frac{1}{n^{l+\tau/2}} \sum_{\nu > k(n)} \nu^l E(|S_\nu|^\tau 1_{B_\nu}) = o(\delta_n), n \in \hat{\mathbb{N}},$

if $l + \tau/2 \leqslant j/2$, $l \geqslant 0$, $\tau \geqslant 0$, $l, \tau \in \mathbb{R}$

(P5) $\frac{1}{n^{l+\tau/2}} \sum_{\nu \leqslant k(n)} \nu^l E(|S_\nu|^\tau 1_{B_\nu}) = o(\delta_n), n \in \hat{\mathbb{N}},$

if $l + \tau/2 \geqslant (j+1)/2$, $l \geqslant 0$, $0 \leqslant \tau \leqslant j$, $l, \tau \in \mathbb{R}$

(P6) $\sum_{\nu \leqslant k(n)} \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu) \simeq \tilde{c}\delta_n, n \in \hat{\mathbb{N}},$

with suitable $\tilde{c} > 0$.

Let us first see whether (P1)–(P6) lead to an example of the desired kind. Put $B = \sum_{\nu \in \mathbb{N}} B_\nu$. Then by (P2)

$$d(B, \sigma(X_1, ..., X_n)) \leqslant \sum_{\nu > n} P(B_\nu) = O\left(\frac{1}{n^{(r-2)/2}} (\lg n)^\beta\right),$$

i.e., $(*)$ is fulfilled. By (P3) we obtain

$$P(S_n^* \leqslant t_0, B) - \Phi(t_0) P(B)$$
$$= \sum_{\nu \leqslant k(n)} (P(S_n^* \leqslant t_0, B_\nu) - \Phi(t_0) P(B_\nu)) + o(\delta_n), \qquad n \in \hat{\mathbb{N}}.$$

Hence, using (P1), Lemma 3 implies that

$$P(S_n^* \leqslant t_0, B) - \Phi(t_0) P(B)$$
$$= \sum_{i=1}^{j} \frac{\Phi^{(i)}(t_0)}{i!} \sum_{\nu \leqslant k(n)} \int_{B_\nu} \left(t_0 f\left(\frac{\nu}{n}\right) - \frac{S_\nu}{\sqrt{n-\nu}}\right)^i dP + o(\delta_n) + \tilde{\varepsilon}_n, \quad (1)$$

where by (P6),

$$\tilde{c}_1 \delta_n \leqslant \tilde{\varepsilon}_n = \sum_{\nu \leqslant k(n)} \varepsilon_{n,\nu} \leqslant \tilde{c}_2 \delta_n, \qquad n \in \hat{\mathbb{N}} \text{ large enough,} \tag{2}$$

with suitable $\tilde{c}_1, \tilde{c}_2 < 0$.

By similar methods as in the proof of Theorem 1 (where (A) and (B) implied (16)) we obtain from (P4), (P5) that there exist $a_1, ..., a_j \in \mathbb{R}$ such that

$$\sum_{i=1}^{j} \frac{\Phi^{(i)}(t_0)}{i!} \sum_{\nu \leqslant k(n)} \int_{B_\nu} \left(t_0 f\left(\frac{\nu}{n}\right) - \frac{S_\nu}{\sqrt{n-\nu}} \right)^i dP$$
$$= \sum_{i=1}^{j} \frac{a_i}{n^{i/2}} + o(\delta_n), \qquad n \in \hat{\mathbb{N}}. \tag{3}$$

Now (1)–(3) imply that

$$P(S_n^* \leqslant t_0, B) = \Phi(t_0)\, P(B) + \sum_{i=1}^{j} \frac{a_i}{n^{i/2}} + \varepsilon_n, \qquad n \in \hat{\mathbb{N}}, \tag{4}$$

where with suitable $c_3, c_4 < 0$,

$$c_3 \delta_n \leqslant \varepsilon_n \leqslant c_4 \delta_n \qquad \text{for sufficiently large } n \in \hat{\mathbb{N}}. \tag{5}$$

By Theorem 1 we obtain

$$P(S_n^* \leqslant t_0, B) = \Phi(t_0)\, P(B) + \varphi(t_0) \sum_{i=1}^{j} \frac{1}{n^{i/2}} Q_{i,B}(t_0) + O(\delta_n). \tag{6}$$

Now (4)–(6) yield $a_i = \varphi(t_0)\, Q_{i,B}(t_0)$, $i = 1, ..., j$, and hence (4), (5) imply the assertion.

Thus it remains to construct $\hat{\mathbb{N}} \subset \mathbb{N}$ and $B_\nu \in \sigma(X_1, ..., X_\nu)$, $\nu \in \mathbb{N}$, disjoint, fulfilling (P1)–(P6). We distinguish the cases $r \in \mathbb{N}$ and $r \notin \mathbb{N}$.

Case $r \in \mathbb{N}$. Here $j = j(r) = r - 3$ and $\beta \geqslant -r/2$. Since

$$P\{\sqrt{\lg \nu}/2 \leqslant S_\nu^* \leqslant \sqrt{\lg \nu}\} = \Phi(\sqrt{\lg \nu}) - \Phi(\sqrt{\lg \nu}/2) \geqslant \frac{1}{\nu^{1/4}}$$

for all sufficiently large ν, there exist $\nu_0 \in \mathbb{N}$ and disjoint $B_\nu \in \sigma(X_1, ..., X_\nu)$, $\nu \geqslant \nu_0$, such that

$$B_\nu \subset \{\sqrt{\lg \nu}/2 \leqslant S_\nu^* \leqslant \sqrt{\lg \nu}\}, \qquad \nu \geqslant \nu_0, \tag{7}$$

$$P(B_\nu) = \frac{1}{\nu^{r/2}} (\lg \nu)^\beta, \qquad \nu \geqslant \nu_0. \tag{8}$$

Put $B_\nu = \varnothing$ for $\nu < \nu_0$ and take $\hat{\mathbb{N}} = \mathbb{N}$. Then obviously (P1), (P2) are fulfilled.

Ad (P3). For sufficiently large n we have by (P2) that

$$\sum_{\nu > k(n)} P(B_\nu) \underset{(\mathrm{P2})}{\leqslant} c \frac{1}{(k(n))^{(r-2)/2}} (\lg k(n))^\beta$$
$$\leqslant c \frac{1}{n^{(r-2)/2}} (\lg n)^{\beta + (r-2)/2} = o(\delta_n).$$

Ad (P4). Let $l + \tau/2 \leqslant j/2 = (r-3)/2$. Then we obtain

$$H(n) := \frac{1}{n^{l+\tau/2}} \sum_{\nu > k(n)} \nu^l E(|S_\nu|^\tau 1_{B_\nu})$$
$$\underset{(7)}{\leqslant} \frac{1}{n^{l+\tau/2}} \sum_{\nu > k(n)} \nu^l (\nu \lg \nu)^{\tau/2} P(B_\nu)$$
$$\underset{(8)}{=} \frac{1}{n^{l+\tau/2}} \sum_{\nu > k(n)} \nu^{l+\tau/2-r/2} (\lg \nu)^{\beta+\tau/2}$$

and $l + \tau/2 - r/2 \leqslant -\frac{3}{2}$ implies

$$H(n) \leqslant c \frac{1}{n^{l+\tau/2}} (k(n))^{l+\tau/2-r/2+1} (\lg k(n))^{\beta+\tau/2}$$
$$\leqslant c \frac{1}{n^{(r-2)/2}} (\lg n)^{\beta + (r-2)/2 - l} = o(\delta_n).$$

Ad (P5). Let $l + \tau/2 \geqslant (j+1)/2 = (r-2)/2$, $0 \leqslant \tau \leqslant j$. Then

$$L(n) = \frac{1}{n^{l+\tau/2}} \sum_{\nu \leqslant k(n)} \nu^l E(|S_\nu|^\tau 1_{B_\nu})$$
$$\underset{(7),(8)}{\leqslant} c \frac{1}{n^{l+\tau/2}} \sum_{2 \leqslant \nu \leqslant k(n)} \nu^{l+\tau/2-r/2} (\lg \nu)^{\beta+\tau/2}.$$

First let $l + \tau/2 = (r-2)/2$. Since $\tau \leqslant j = r-3$ this implies $l \geqslant \frac{1}{2}$ and hence by a simple calculation

$$L(n) \leqslant c \frac{1}{n^{(r-2)/2}} \sum_{2 \leqslant \nu \leqslant k(n)} \frac{1}{\nu} (\lg \nu)^{-1/2+\beta+(r-2)/2}$$
$$= \left\{ \begin{array}{ll} o\left(\dfrac{\lg \lg n}{n^{(r-2)/2}}\right) : \beta = -r/2 \\ o\left(\dfrac{(\lg n)^{\beta+r/2}}{n^{(r-2)/2}}\right) : \beta > -r/2 \end{array} \right\} = o(\delta_n).$$

It remains to consider the case $l+\tau/2>(r-2)/2$. Then $l+\tau/2-r/2>-$ and we have

$$L(n)\leqslant c\frac{1}{n^{l+\tau/2}}((k(n))^{l+\tau/2-r/2+1}(\lg k(n))^{\beta+\tau/2})$$
$$\leqslant c\frac{1}{n^{(r-2)/2}}(\lg n)^{\beta+(r-2)/2}=o(\delta_n).$$

Ad (P6). We have by (8)

$$\sum_{\nu\leqslant k(n)}\left(\frac{\nu\lg\nu}{n}\right)^{(j+1)/2}P(B_\nu)$$
$$\underset{(8)}{=}\frac{1}{n^{(r-2)/2}}\sum_{\nu_0\leqslant\nu\leqslant k(n)}\frac{1}{\nu}(\lg\nu)^{\beta+(r-2)/2}$$
$$\simeq\left.\begin{cases}\dfrac{\lg\lg n}{n^{(r-2)/2}}, & \text{if}\quad\beta=-r/2\\[2ex] \dfrac{1}{\beta+r/2}\dfrac{(\lg n)^{\beta+r/2}}{n^{(r-2)/2}}, & \text{if}\quad\beta>-r/2\end{cases}\right\}=\tilde{c}\delta_n.$$

Case $r\notin\mathbb{N}$. Here $j=j(r)=[r]-2$ and $\beta\geqslant-(r-2)/2$. Put

$$\tilde{\mathbb{N}}:=\{2^{2^i}:i\in\mathbb{N}\}\qquad\text{and}\qquad\hat{\mathbb{N}}:=\left\{n\in\mathbb{N}:k(n)=\left[c_0\frac{n}{\lg n}\right]\in\tilde{\mathbb{N}}\right\}.$$

Then there exist $\nu_0\in\mathbb{N}$ and disjoint $B_\nu\in\sigma(X_1,...,X_\nu)$, $\nu\in\tilde{\mathbb{N}}$, $\nu\geqslant\nu_0$, such that

$$B_\nu\subset\{\sqrt{\lg\nu}/2\leqslant S_\nu^*\leqslant\sqrt{\lg\nu}\}\tag{9}$$

$$P(B_\nu)=\frac{1}{\nu^{(r-2)/2}}(\lg\nu)^\beta,\qquad\nu\in\tilde{\mathbb{N}},\ \nu\geqslant\nu_0.\tag{10}$$

Put $B_\nu=\varnothing$ if $\nu<\nu_0$ or $\nu\notin\tilde{\mathbb{N}}$. Then obviously (P1), (P2) are fulfilled.

Ad (P3). Let $n\in\hat{\mathbb{N}}$. Then $k(n)\in\tilde{\mathbb{N}}$ and therefore $B_\nu=\varnothing$ if $k(n)<\nu<k^2(n)$. Hence we obtain for sufficiently large $n\in\hat{\mathbb{N}}$

$$\sum_{\nu>k(n)}P(B_\nu)=\sum_{\nu>n}P(B_\nu)=o(\delta_n),\qquad n\in\hat{\mathbb{N}}.$$

Ad (P4). Let $l+\tau/2\leqslant j/2=([r]-2)/2$. We have by (9), (10) that

$$H(n)=\frac{1}{n^{l+\tau/2}}\sum_{\nu>k(n)}\nu^lE(|S_\nu|^\tau 1_{B_\nu})$$
$$\underset{(9)}{\leqslant}\frac{1}{n^{l+\tau/2}}\sum_{\nu>k(n)}\nu^{l+\tau/2}(\lg\nu)^{\tau/2}P(B_\nu)$$
$$\underset{(10)}{\leqslant}\frac{1}{n^{l+\tau/2}}\sum_{\nu>k(n),\,\nu\in\tilde{\mathbb{N}}}\nu^{l+\tau/2-(r-2)/2}(\lg\nu)^{\tau/2+\beta}.$$

Let $n \in \hat{\mathbb{N}}$. Then $\nu > k(n)$, $\nu \in \tilde{\mathbb{N}}$, implies $\nu \geqslant k^2(n) \geqslant k(n) \lg k(n)$. As $l+\tau/2-(r-2)/2<0$ we consequently obtain for sufficiently large $n \in \hat{\mathbb{N}}$

$$\begin{aligned} H(n) &\leqslant c \frac{1}{n^{l+\tau/2}} (k(n) \lg k(n))^{l+\tau/2-(r-2)/2} (\lg n)^{\beta+\tau/2} \\ &\leqslant c \frac{1}{n^{(r-2)/2}} (\lg n)^{\beta+\tau/2} \\ &= o\left(\frac{1}{n^{(r-2)/2}} (\lg n)^{\beta+(r-2)/2}\right) = o(\delta_n), \qquad n \in \hat{\mathbb{N}}. \end{aligned}$$

Ad (P5). Let $l+\tau/2 \geqslant (j+1)/2 = ([r]-1)/2$ and $0 \leqslant \tau \leqslant j$. We have

$$\begin{aligned} L(n) \; &:= \; \frac{1}{n^{l+\tau/2}} \sum_{\nu \leqslant k(n)} \nu^l E(|S_\nu|^\tau 1_{B_\nu}) \\ &\underset{(9),\,(10)}{\leqslant} \frac{1}{n^{l+\tau/2}} \sum_{\nu_0 \leqslant \nu \leqslant k(n),\, \nu \in \tilde{\mathbb{N}}} \nu^{l+\tau/2-(r-2)/2} (\lg \nu)^{\beta+\tau/2}. \end{aligned}$$

As $l+\tau/2 \geqslant ([r]-1)/2 > (r-2)/2$ and as $k(n) \in \tilde{\mathbb{N}}$ for all $n \in \hat{\mathbb{N}}$, we obtain for all sufficiently large $n \in \hat{\mathbb{N}}$

$$\begin{aligned} L(n) &\leqslant c \frac{1}{n^{l+\tau/2}} (k(n))^{l+\tau/2-(r-2)/2} (\lg k(n))^{\beta+\tau/2} \\ &\leqslant c \frac{1}{n^{(r-2)/2}} (\lg n)^{\beta+(r-2)/2-l}. \end{aligned}$$

As $l+\tau/2 > j/2$ and $\tau \leqslant j$, we have $l > 0$. Therefore

$$L(n) = o\left(\frac{1}{n^{(r-2)/2}} (\lg n)^{\beta+(r-2)/2}\right) = o(\delta_n), \qquad n \in \hat{\mathbb{N}}.$$

Ad (P6). Since $j+1 > r-2$, we obtain by (10) for all $n \in \hat{\mathbb{N}}$

$$\begin{aligned} &\sum_{\nu \leqslant k(n)} \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu) \\ &\quad \underset{(10)}{=} \frac{1}{n^{(j+1)/2}} \sum_{\nu_0 \leqslant \nu \leqslant k(n),\, \nu \in \tilde{\mathbb{N}}} \nu^{(j+1)/2-(r-2)/2} (\lg \nu)^{\beta+(j+1)/2} \\ &\quad \simeq \frac{1}{n^{(j+1)/2}} (k(n))^{(j+1)/2-(r-2)/2} (\lg k(n))^{\beta+(j+1)/2} \\ &\quad \simeq \tilde{c} \frac{1}{n^{(r-2)/2}} (\lg n)^{\beta+(r-2)/2} = \tilde{c}\delta_n, \qquad n \in \hat{\mathbb{N}} \end{aligned}$$

with $\tilde{c} := c_0^{(j+1)/2-(r-2)/2}$.

LEMMA 3. *Let X_n, $n \in \mathbb{N}$, be i.i.d. $N(0, 1)$-distributed. Let $j \in \mathbb{N}$ and put $f(x) = (1-x)^{-1/2} - 1$.*

Then there exist $t_0 \in \mathbb{R}$, $c_0 \in (0, 1]$ such that for all sufficiently large $n \in \mathbb{N}$, all $\nu \leqslant c_0\, n/\lg n$, and all $B_\nu \in \sigma(X_1, ..., X_\nu)$ with $B_\nu \subset \{\sqrt{\lg \nu}/2 \leqslant S_\nu^* \leqslant \sqrt{\lg \nu}\}$,

$$\begin{aligned} &P(S_n^* \leqslant t_0, B_\nu) - \Phi(t_0)\, P(B_\nu) \\ &\quad = \sum_{i=1}^{j} \frac{\Phi^{(i)}(t_0)}{i!} \int_{B_\nu} \left(t_0 f\left(\frac{\nu}{n}\right) - \frac{S_\nu}{\sqrt{n-\nu}} \right)^i dP + \varepsilon_{n,\nu} \end{aligned}$$

holds, where for suitable $c_1, c_2 < 0$,

$$c_1 \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu) \leqslant \varepsilon_{n,\nu} \leqslant c_2 \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu).$$

Proof. It is easy to see that there exists $t_0 \geqslant 1$ with

$$(-1)^{j+1}\, \Phi^{(j+1)}(t_0) < 0. \tag{1}$$

Since $\omega \to \Phi(t_0 \sqrt{n/(n-\nu)} - S_\nu(\omega)/\sqrt{n-\nu})$ is a version of $P(S_n^* \leqslant t_0 \mid X_1, ..., X_\nu)$, $\nu < n$, and since $B_\nu \in \sigma(X_1, ..., X_\nu)$ we obtain

$$\begin{aligned} &P(S_n^* \leqslant t_0, B_\nu) - \Phi(t_0)\, P(B_\nu) \\ &\quad = \int_{B_\nu} \left(\Phi\left(t_0 \sqrt{\frac{n}{n-\nu}} - \frac{S_\nu}{\sqrt{n-\nu}} \right) - \Phi(t_0) \right) dP. \end{aligned} \tag{2}$$

By the Taylor expansion we have

$$\begin{aligned} &\Phi(u) - \Phi(t_0) \\ &\quad = \sum_{i=1}^{j} \frac{\Phi^{(i)}(t_0)}{i!} (u - t_0)^i + \frac{1}{(j+1)!} (u - t_0)^{j+1}\, \Phi^{(j+1)}(\xi) \end{aligned} \tag{3}$$

with $\xi \in [u, t_0]$. Put $u = u_{\nu,n}(\omega) = t_0 \sqrt{n/(n-\nu)} - (1/\sqrt{n-\nu})\, S_\nu(\omega)$; then

$$u - t_0 = t_0 f\left(\frac{\nu}{n}\right) - \frac{S_\nu}{\sqrt{n-\nu}}. \tag{4}$$

Hence (2)–(4) imply the assertion if we prove that the stated inequality for $\varepsilon_{n,\nu}$ is fulfilled with

$$\begin{aligned} \varepsilon_{n,\nu} &= \frac{1}{(j+1)!} \int_{B_\nu} (u - t_0)^{j+1}\, \Phi^{(j+1)}(\xi)\, dP \\ &= \frac{1}{(j+1)!} \sum_{l=0}^{j+1} \binom{j+1}{l} \int_{B_\nu} \left(t_0 f\left(\frac{\nu}{n}\right) \right)^l \\ &\qquad \times (-1)^{j+1-l} \left(\frac{S_\nu}{\sqrt{n-\nu}} \right)^{j+1-l} \Phi^{(j+1)}(\xi)\, dP, \end{aligned}$$

where $\xi = \xi_{\nu,n}(\omega) \in [u_{\nu,n}(\omega), t_0]$. As $S_\nu(\omega) \leqslant \sqrt{\nu \lg \nu}$ for each $\omega \in B_\nu$ we obtain for all $1 \leqslant l \leqslant j+1$, $\nu \leqslant n/\lg n$

$$\left|\int_{B_\nu} \left(t_0 f\left(\frac{\nu}{n}\right)\right)^l \left(\frac{S_\nu}{\sqrt{n-\nu}}\right)^{j+1-l} \Phi^{(j+1)}(\xi)\, dP\right|$$

$$\leqslant c \left(\frac{\nu}{n}\right)^l \frac{1}{n^{(j+1-l)/2}} \int_{B_\nu} |S_\nu|^{j+1-l}\, dP$$

$$\leqslant c \frac{1}{n^{(j+1)/2}} \frac{\nu^l}{n^{l/2}} (\nu \lg \nu)^{(j+1-l)/2} P(B_\nu)$$

$$\leqslant c \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu) \left(\frac{\nu}{n}\right)^{l/2}$$

$$\leqslant c \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu) \left(\frac{1}{\lg n}\right)^{l/2}$$

Hence the stated inequality for $\varepsilon_{n,\nu}$ holds, if there exist $0 < c_0 \leqslant 1$ and $c_3, c_4 < 0$ such that for all sufficiently large n and all $\nu \leqslant c_0(n/\lg n)$,

$$c_3 \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu)$$

$$\leqslant \int_{B_\nu} \left(\frac{S_\nu}{\sqrt{n-\nu}}\right)^{j+1} (-1)^{j+1} \Phi^{(j+1)}(\xi)\, dP \leqslant c_4 \left(\frac{\nu \lg \nu}{n}\right)^{(j+1)/2} P(B_\nu). \tag{5}$$

To prove (5) choose $\delta_0 > 0$ and $c_5, c_6 < 0$ such that

$$c_5 \leqslant (-1)^{j+1} \Phi^{(j+1)}(\xi) \leqslant c_6 \qquad \text{for all } \xi \in [t_0 - \delta_0, t_0 + \delta_0]. \tag{6}$$

This is possible according to (1). As $B_\nu \subset \{\sqrt{\lg \nu}/2 \leqslant S_\nu^* \leqslant \sqrt{\lg \nu}\}$ it is easy to see that there exist $c_0 \in (0, 1]$, $n_0 \in \mathbb{N}$ such that

$$u_{\nu,n}(\omega) = t_0 \sqrt{\frac{n}{n-\nu}} - \frac{S_\nu(\omega)}{\sqrt{n-\nu}} \in [t_0 - \delta_0, t_0 + \delta_0]$$

and hence

$$\xi_{\nu,n}(\omega) \in [t_0 - \delta_0, t_0 + \delta_0] \tag{7}$$

for all $\omega \in B_\nu$, $n \geqslant n_0$, and $\nu \leqslant c_0(n/\lg n)$. Now (6) and (7) imply (5). This finishes the proof of the assertion.

LEMMA 4. *Let $X_n \in \mathcal{L}_r$, $n \in \mathbb{N}$, be i.i.d. with $E(X_n) = 0$ and $E(X_n^2) = 1$. Let $r \geqslant 3$; then we have for all $\gamma \geqslant \frac{1}{2}$ and $0 < \tau < r$*

$$E[|S_m|^\tau 1_{\{|S_m^*| \geqslant \sqrt{r-1}\,(\lg m)^\gamma\}}] \leqslant c m^{\tau/2 - (r-2)/2} (\lg m)^{\gamma(\tau - r)}$$

with a suitable constant $c > 0$.

Proof. We have

$$E[|S_m|^\tau 1_{\{|S_m^*| \geq \sqrt{r-1}\,(\lg m)^\gamma\}}]$$

$$= [(m(r-1))^{1/2} (\lg m)^\gamma]^\tau$$

$$\times E\left[\left|\frac{|S_m|}{\sqrt{m(r-1)}\,(\lg m)^\gamma}\right|^\tau 1_{\{|S_m/\sqrt{m(r-1)}\,(\lg m)^\gamma|^\tau \geq 1\}}\right]$$

$$\leqslant cm^{\tau/2} (\lg m)^{\gamma\tau} \sum_{k \in \mathbb{N}} P\left\{\left|\frac{S_m}{\sqrt{m(r-1)}\,(\lg m)^\gamma}\right|^\tau \geq k\right\}$$

$$\leqslant cm^{\tau/2}(\lg m)^{\gamma\tau} \sum_{k \in \mathbb{N}} P\{|S_m^*| \geq k^{1/\tau}\sqrt{r-1}\,(\lg m)^\gamma\}$$

$$\underset{(*)}{\leqslant} cm^{\tau/2}(\lg m)^{\gamma\tau} \sum_{k \in \mathbb{N}} \frac{1}{m^{(r-2)/2}} \frac{1}{k^{r/\tau}(\lg m)^{\gamma r}}$$

$$\leqslant cm^{\tau/2-(r-2)/2}(\lg m)^{\gamma(\tau-r)},$$

where (∗) follows from Theorem 2 of [5] or from Corollary 17.12 of [1].

LEMMA 5. *Let* $\mathbb{N}_1 = \{2^\nu : \nu \in \mathbb{N}\}$ *and* $N_n = \{\nu \in \mathbb{N}_1 : \nu \leqslant n/\lg n\}$. *Then*

$$\sum_{\nu \in N_n} \nu^\varepsilon (\lg \nu)^\gamma = \begin{cases} O(n^\varepsilon(\lg n)^{\gamma-\varepsilon}), & \varepsilon > 0, \quad \gamma \in \mathbb{R} \\ O((\lg n)^{\gamma+1}), & \varepsilon = 0, \quad \gamma > -1 \\ O(\lg\lg n), & \varepsilon = 0, \quad \gamma = -1 \\ O(1), & \varepsilon = 0, \quad \gamma < -1. \end{cases}$$

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